Unfolding of Double-Zero Eigenvalue Bifurcations for Supersonic Flow Past a Pitching Wedge

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In this paper a complete unfolding of a codimension two bifurcation due to a double-zero eigenvalue of the equations of pitching motion of a double-wedge airfoil is carried out. The critical parameter values along with various parameters related to stiffness derivatives and damping derivatives are tabulated for a thin airfoil for a range of Mach numbers. Both local and global bifurcations of the trivial and nontrivial fixed points and uniqueness of limit cycles are determined.

I. Introduction

RECENTLY Hui and Tobak[1] analyzed the Hopf bifurcation that results when the steady flight of an aircraft becomes unstable in pitch by increasing the angle of attack. For the case of a double-wedge airfoil it was found that, in addition to Hopf bifurcation, degenerate Hopf bifurcation can also take place due to the violation of a certain transversality condition.[2] Since such degenerate bifurcation is nongeneric, Sri Namachchivaya and Van Roessel[3] made use of the results of singularity theory to unfold these bifurcations.

The purpose of this paper is to extend these results on the nonlinear analysis of a pitching aircraft at high angles of attack. It will be shown that, in addition to the aforementioned codimension one and two bifurcations, there exist codimension two bifurcations associated with a double-zero eigenvalue. Usually a zero eigenvalue implies a simple bifurcation. In the case of a double-zero eigenvalue with nonsemisimple Jordan form and no further degeneracy, two parameters are required for a complete universal unfolding.[4] All possible bifurcations that take place in the neighborhood of this bifurcation point will be obtained by making use of these unfolding parameters. A family of limit cycles may branch off from the equilibrium surface in the vicinity of such a critical point. For the equations of motion for a pitching wedge such a double-zero eigenvalue does occur at certain critical values of the system parameters. The partial unfolding for this case is carried out below.

Consider an aircraft in steady flight at an angle of attack σ. Suppose some disturbances take place at time $t=0$, e.g., due to a change in the flap deflection angle; the aircraft will subsequently undergo an unsteady motion relative to its steady flight. Such an unsteady motion of the aircraft modifies the air flow and hence the aerodynamic forces on the aircraft, which, in turn, determine its motion. Thus the aircraft's subsequent motion can only be determined by simultaneously solving the unsteady flow equations of the air and the equations of motion of the vehicle as a rigid body, aeroelastic effects being assumed negligible.

Although simultaneously solving the coupled equations in principle represents an exact approach to the problem of arbitrary maneuvers, it is inevitably a very difficult and costly approach. In classical aerodynamics, the traditional approximate approach is to assume the pitching motion to be a small amplitude periodic oscillation consisting of simple harmonics. On this basis the flow equations are decoupled from the inertia equation and are linearized to determine the aerodynamic response to such a harmonic motion. The so-called aerodynamic coefficients thus obtained are then used to predict the motion of the aircraft. Even though this approach ignores the time-history effects on the flowfield and the aircraft motion, it gives a good approximation for calculating the aerodynamic response from the unsteady flow equations and hence the pitching moment. This approximation, which has been adopted by Hui and Tobak[1] and Sri Namachchivaya and Van Roessel[3] in their investigations of this problem, is used in this paper.

II. Statement of the Problem

Consider an aircraft in flight, free to undergo a single-degree-of-freedom pitching motion. The equations of pitching motion can be expressed as

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t}=\dot{\alpha}, \qquad I\frac{\mathrm{d}\dot{\alpha}}{\mathrm{d}t}=M(t) \tag{1}$$

where α is the instantaneous angle of attack, I the moment of inertia of the vehicle about the pivot axis, and $M(t)$ the pitching moment at instantaneous time t of the aerodynamic forces about the same axis. When the motion is slowly varying,[5] the pitching moment $M(t)$ may be characterized with sufficient accuracy by the instantaneous angle of attack $\alpha(t)$ and the instantaneous rate of change of the angle of attack $\dot{\alpha}(t)$. Suppose $\alpha=\sigma$ is an equilibrium state of the system of Eqs. (1); then, putting $\alpha(t)=\sigma+\psi(t)$, the variational equations about the equilibrium position can be written as

$$\frac{\mathrm{d}\psi}{\mathrm{d}t}=\dot{\psi}, \qquad \frac{\mathrm{d}\dot{\psi}}{\mathrm{d}t}=M(t) \tag{2}$$

where ψ is the angular displacement of motion measured from the angle of attack σ of the steady flight. It is assumed that the moment required to trim the aircraft at σ has been accounted for, so that $M(t)$ is a measure of the perturbation moment only and is determined from the instantaneous surface pressure. As noted earlier, following the mathematical modeling

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approach of Tobak and Schiff,[5] instantaneous pitching moment can be given as

$$M(t)=\frac{\rho_\infty V_\infty^2}{2}\bar{S}L\,[C_m(0,0,\sigma,h)-C_m(\psi,\dot\psi,\sigma,h)]$$

where ρ_∞ and V_∞ are the freestream density and velocity, respectively; $\bar{S}$ and L are the reference area and length; and h represents the distance between the apex and the pivot position as defined in Fig. 1. The function $C_m(\psi,\dot\psi,\sigma,h)$ represents the pitching moment coefficient of the aerodynamic forces about the pivot axis and $C_m(0,0,\sigma,h)$ is its steady value at a fixed angle of attack σ. Even though C_m depends on the flight Mach number M_∞, the specific heat of the air and the aircraft shape, these parameters will be considered as "passive" parameters in this analysis. For a *finite amplitude, slow, pitching motion* with angular displacement $\psi(t)$ around a mean angle of attack σ, with terms of $0(\dot\psi^2,\ddot\psi)$ assumed negligible, we can write[1]

$$-C_m(\psi,\dot\psi,\sigma,h)=f(\sigma+\psi,h)+g(\sigma+\psi,h)\dot\psi$$

which reduces the second of Eqs. (2) to

$$\frac{\mathrm{d}\dot\psi}{\mathrm{d}t}=F(\psi,\dot\psi,\sigma,h)$$

where

$$F(\psi,\dot\psi,\sigma,h)=\frac{M(t)}{I}=\kappa[f(\sigma+\psi,h)-f(\sigma,h)+g(\sigma+\psi,h)\dot\psi]$$

$$\kappa=\frac{1}{2I}\rho_\infty V_\infty^2\bar{S}L$$

Equations (2) represent a pair of autonomous differential equations in R^2, the trivial solution of which is $\psi=0$. The objective of this investigation is to understand the stability and the bifurcation behavior of the stationary solutions of Eqs. (2) as the system parameters σ and h are varied.

III. Bifurcation of Fixed Points

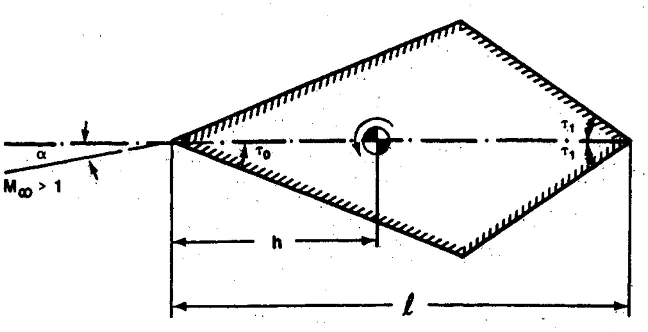
The functions $f(\sigma,h)$ and $g(\sigma,h)$ are related to the stiffness derivative $S(\sigma,h)$ and the damping derivative $D(\sigma,h)$ of classical aerodynamics as follows:

$$\kappa\frac{\partial f}{\partial\sigma}(\sigma,h)=-S(\sigma,h),\qquad \kappa g(\sigma,h)=-D(\sigma,h)$$

Introducing new state variables $\bar{x}=\psi$, $\bar{y}=\dot\psi$, Eqs. (2) may be written in the form

$$\bar{x}'=\bar{y} \tag{3a}$$

$$\bar{y}'=\bar\mu_1\bar{x}+\bar\mu_2\bar{y}+\bar{p}_0\bar{x}^2+\bar{q}_0\bar{x}\bar{y}+\bar{p}_1\bar{x}^3+\bar{q}_1\bar{x}^2\bar{y} \tag{3b}$$



, Aerofoil at Angle of Attack α

Fig. 1 Thin double-edge airfoil.

where

$$\bar\mu_1=-S(\sigma,h),\qquad \bar\mu_2=-D(\sigma,h),\qquad \bar{p}_0=-\frac{1}{2}\frac{\partial S}{\partial\sigma}(\sigma,h)$$

$$\bar{q}_0=-\frac{\partial D}{\partial\sigma}(\sigma,h),\quad \bar{p}_1=-\frac{1}{3!}\frac{\partial^2 S}{\partial\sigma^2}(\sigma,h),\quad \bar{q}_1=-\frac{1}{2}\frac{\partial^2 D}{\partial\sigma^2}(\sigma,h)$$

It is evident that the nongeneric case $\bar\mu_1=\bar\mu_2=0$, i.e., $D(\sigma_c,h_c)=0$ and $S(\sigma_c,h_c)=0$, gives rise to a double-zero eigenvalue with nonsemisimple Jordan form as the Jacobian. It is well known that the damping and the stiffness derivatives are respectively quadratic and linear in h, i.e.,

$$D(\sigma,h)=D_0(\sigma)+D_1(\sigma)h+D_2(\sigma)h^2$$

$$S(\sigma,h)=S_0(\sigma)+S_1(\sigma)h$$

Furthermore, the qualitative variations of the quantities $D(\sigma,h)$ and $S(\sigma,h)$ with σ and h can be found in Hui[6] for a double-wedge aerofoil. The variations of the components of S and D, namely S_0, S_1, D_0, D_1, and D_2 are given graphically in Sri Namachchivaya and Van Roessel.[3] The critical values of σ and h are obtained by letting $D=S=0$ in the preceding expressions. The critical parameter values and the various coefficients needed for the analysis are given in Table 1 for $\tau_0=\tau_1=5$ deg. Introducing new variables x, y, and new time t,

$$x=\left(\frac{\bar{q}_0^2}{\bar{p}_0}\right)\bar{x},\qquad y=\left(\frac{\bar{q}_0^3}{\bar{p}_0^2}\right)\bar{y},\qquad t=\left(\frac{p_0}{q_0}\right)\bar{t},\qquad p_0,q_0\neq 0$$

yields

$$\dot{x}=y \tag{4a}$$

$$\dot{y}=\mu_1x+\mu_2y+x^2+xy+p_1x^3+q_1x^2y \tag{4b}$$

where

$$\mu_1=\left(\frac{\bar{q}_0}{\bar{p}_0}\right)\bar\mu_1,\qquad \mu_2=\left(\frac{\bar{q}_0}{\bar{p}_0}\right)\bar\mu_2,\qquad p_1=\frac{\bar{p}_1}{\bar{q}_0^2},\qquad q_1=\frac{\bar{q}_1\bar{p}_0}{\bar{q}_0^3}$$

The theory of normal forms deals with finding near-identity coordinate transformations, which simplify the analytic expressions of the nonlinear terms. The resulting simplified nonlinear equations are said to be in normal form. Equations (4) are in normal form since the expression for the normal form, for a nonlinear system with quadratic and cubic nonlinearities and a double-zero nonsemisimple Jordan block, is identical to that of Eqs. (4). Furthermore, when the quadratic nonlinearities are not identically zero, the higher-order terms (i.e., cubic terms) do not contribute to qualitative changes in the phase portrait. Thus, a simplified set of equations

$$\dot{x}=y \tag{5a}$$

$$\dot{y}=\mu_1x+\mu_2y+x^2+xy \tag{5b}$$

which reveals all the principal phenomena contained in the general problem, will be analyzed. In Eqs. (5), μ_1 and μ_2 are the unfolding parameters and are related to the determinant and the trace respectively of the linear operator of Eq. (3). We first seek the fixed points of Eqs. (5), which are given by $(x_0,y_0)=(0,0)$ and $(x_0,y_0)=(-\mu_1,0)$. Putting $x=x_0+u$ and $y=y_0+v$, the variational equations about the fixed point can be written as

$$\dot{u}=v \tag{6a}$$

$$\dot{v}=\alpha_1u+\alpha_2v+u^2+uv \tag{6b}$$

where $\alpha_1=\mu_1+2x_0$ and $\alpha_2=\mu_2+x_0$. The eigenvalues of the fixed point are given by

$$\lambda_{1,2}=(\alpha_2/2)\pm\sqrt{\alpha},\qquad \alpha=(\alpha_2^2/4)+\alpha_1 \tag{7}$$

Table 1 Critical parameter values associated with double-zero eigenvalue

M_∞	σ_c	h_c	$-(\partial S/\partial\sigma)$	$-(\partial D/\partial\sigma)$	$-(\partial^2 S/\partial^2\sigma)$	$-(\partial^2 D/\partial^2\sigma)$
2.0000	12.3886	0.4432	0.3158	1.5078	10.5616	25.1282
2.5000	19.2534	0.4471	0.1738	1.6417	9.5587	29.1733
3.0000	23.7198	0.4524	0.1240	1.7658	9.9810	33.3559
3.5000	26.7177	0.4571	0.1166	1.8872	10.8663	37.2688
4.0000	28.7890	0.4608	0.1233	1.9906	11.7336	40.5696
5.0000	31.3435	0.4656	0.1408	2.1382	12.9768	45.3055
6.0000	32.7784	0.4684	0.1506	2.2277	13.6656	48.2253
7.0000	33.6578	0.4701	0.1543	2.2831	14.0607	50.0638
8.0000	34.2344	0.4713	0.1557	2.3190	14.3154	51.2766
9.0000	34.6325	0.4721	0.1565	2.3433	14.4953	52.1131
10.0000	34.9187	0.4727	0.1571	2.3604	14.6224	52.7144

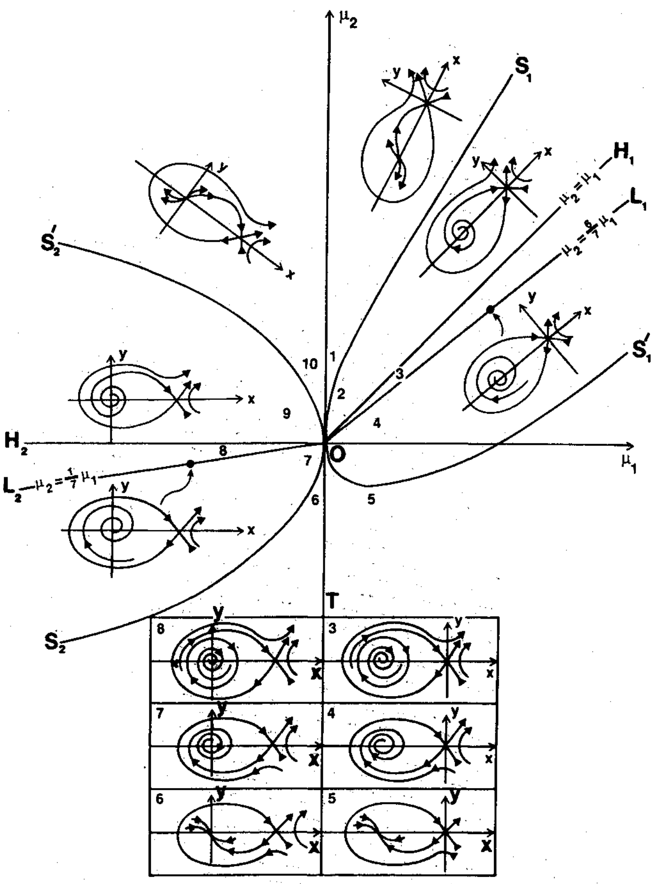


Fig. 2 Various bifurcations and phase portraits of Eq. (5).

It is evident from Eq. (7) that the fixed point is asymptotically stable if $\alpha_1<0$ and $\alpha_2<0$ and goes through a Hopf bifurcation at $\alpha_2=0$ and $\alpha_1<0$. Thus, making use of the transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \sqrt{-\alpha_1} & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

Eq. (6) in the neighborhood of $\alpha_2=0$ can be written as

$$\dot{\xi} = \alpha_2\xi - \sqrt{-\alpha_1}\eta + \frac{1}{\sqrt{-\alpha_1}}\eta^2 + \eta\xi \tag{8a}$$

$$\dot{\eta} = \sqrt{-\alpha}\xi \tag{8b}$$

Now the formulas for Hopf bifurcation given by Guckenheimer and Holmes[7] are used to obtain the equation governing the bifurcating path as

$$2Ra^3 + \alpha_2 a = 0, \qquad R = -(1/8\alpha_1)$$

where a represents the amplitude of the bifurcating periodic solution. Since $\alpha_1<0$, it is obvious that the fixed points undergo an unstable subcritical Hopf bifurcation when $\alpha_2=0$. Moreover the fixed point goes through a simple transcritical bifurcation at $\alpha_1=0$ and $\alpha_1\neq 0$. It may be noted that the fixed point, a stable node for $\alpha_2<0$ and an unstable node or $\alpha_2>0$, becomes a saddle-node at $\alpha_1=0$ while undergoing a transcritical bifurcation.

IV. Global Bifurcations

It is clear from the analysis performed thus far and Fig. 2, that the phase portraits in regions 3 and 4 are not homeomorphic, since the former has a limit cycle and the latter does not. For similar reasons, regions 7 and 8 also are not homeomorphic. Hence, there must be additional global bifurcations occurring in which the nature of the fixed points do not change, but the phase portraits as a whole undergo a topological change. In this section, such global bifurcations are examined. The fixed point is a saddle point when $\alpha_1>0$, and making use of the transformation

$$u=\epsilon^2 z_1, \qquad v=\epsilon^3 z_2, \qquad \alpha_1=\epsilon^2\nu_1, \qquad \alpha_2=\epsilon^2\nu_2, \qquad \tau=\epsilon t \tag{9}$$

yields

$$\frac{\mathrm{d}z_1}{\mathrm{d}\tau} = z_2 \tag{10a}$$

$$\frac{\mathrm{d}z_2}{\mathrm{d}\tau} = \nu_1 z_1 + z_1^2 + \epsilon(\nu_2 z_2 + z_1 z_2) \tag{10b}$$

where $|\epsilon|\ll 1$ and $\epsilon\leq 0$. For $\epsilon\to 0$ the preceding equations become an integrable Hamiltonian system with Hamiltonian

$$H(z_1,z_2) = \frac{z_2^2}{2} - \nu_1\frac{z_1^2}{2} - \frac{z_1^3}{3}$$

and $(z_1=z_2=0)$ is a stable point and possesses a "saddle connection." Since the Hamiltonian is conserved, the level curves $H=\text{const}$ are solutions of Eqs. (10) with $\epsilon=0$. Furthermore, the value of the Hamiltonian at the saddle point is $H(0,0)=0$, and the points of intersections of the saddle loop with the axis $z_2=0$ are $z_1=0$ and $z_1=-(3/2)\nu_1$. The unperturbed trajectories of the saddle-loop at $z_1=-(3/2)\nu_1$ can be obtained as

$$z_1(t,t_0) = -\frac{3\nu_1}{2}\operatorname{sech}^2\left[\frac{\sqrt{\nu_1}}{2}(t-t_0)\right] \tag{11a}$$

$$z_2(t,t_0) = \frac{3{\nu_1}^{3/2}}{2}\operatorname{sech}^2\left[\frac{\sqrt{\nu_1}}{2}(t-t_0)\right]\tanh\left[\frac{\sqrt{\nu_1}}{2}(t-t_0)\right] \tag{11b}$$

Following the Melnikov procedure given in Guckenheimer and Holmes[7] for the perturbed autonomous system (10) (with $\epsilon \neq 0$), we obtain the condition that

$$\int_{-\infty}^{\infty} v^2(t,t_0)\,[\nu_2+u(t,t_0)]\,\mathrm{d}t=0 \tag{12}$$

for the saddle connection not to break under perturbation. Equation (12) may be written as

$$\frac{27}{4}(\nu_1)^{7/2}(I_1-I_2)=0$$

where

$$I_1=\frac{2\nu_2}{3\nu_1}\int_{-\infty}^{\infty}\operatorname{sech}^4\xi\,\tanh^2\xi\,\mathrm{d}\xi=\frac{8\nu_2}{45\nu_1}$$

$$I_2=\int_{-\infty}^{\infty}\operatorname{sech}^6\xi\,\tanh^2\xi\,\mathrm{d}\xi=\frac{16}{105}$$

Thus the saddle connection is preserved when

$$\nu_2=\frac{6}{7}\nu_1 \qquad \text{or} \qquad \alpha_2=\frac{6}{7}\alpha_1$$

It can be concluded that there exist two saddle connections: one at $\mu_2=6/7\ \mu_1$ passing through the trivial solution, and one at $\mu_2=1/7\ \mu_1$ that passes through the nontrivial solution as shown in Fig. 1.

The preceding calculations indicate the existence of a limit cycle in regions 3 and 8 in Fig. 2. The uniqueness of this limit cycle will be demonstrated following the procedure outlined in Chow and Hale[8] and Carr et al.[9] Every limit cycle within the saddle-loop must encircle the equilibrium point $(-\nu_1,0)$ crossing the x axis between $-\nu_1$ and 0 at $(b,0)$. Let the other crossing point be $(c,0)$. The limit cycle for the perturbed system are denoted as $\Gamma_\epsilon(b,\nu_1,\nu_2)$. Along the solution of Eq. (10) we have

$$\dot{H}(z_1,z_2)=\epsilon\, z_2^2\,(\nu_2+z_1)$$

and, since $\Gamma_\epsilon(b,\nu_1,\nu_2)$ is a limit cycle, we have

$$\int_{\Gamma_\epsilon}\dot{H}\,\mathrm{d}t=0 \qquad \text{i.e., } F(b,\epsilon,\nu_1,\nu_2)=\int_{\Gamma_\epsilon} z_1^2\,(\nu_2+z_1)\,\mathrm{d}t=0$$

The function $F(b,0,\nu_1,\nu_2)$ may be written explicitly as

$$F(b,0,\nu_1,\nu_2)=\nu_2\bar{J}_0(b,\nu_1)+\bar{J}_1(b,\nu_1) \tag{13}$$

where

$$\bar{J}_0(b,\nu_1)=\int_{\Gamma_0} z_2^2\,\mathrm{d}t, \qquad \bar{J}_1(b,\nu_1)=\int_{\Gamma_0} z_1 z_2^2\,\mathrm{d}t$$

Thus, the solution of $F(b,0,\nu_1,\nu_2)=0$ is given by

$$\nu_2=-\bar{J}_1(b,\nu_1)/\bar{J}_0(b,\nu_1)$$

Differentiating Eq. (13) yields

$$\frac{\partial F}{\partial \nu_2}(b,0,\nu_1,\nu_2)=\bar{J}_0(b,\nu_1)\neq 0$$

which implies, by the implicit function theorem (IFT), that there exists a unique continuously differentiable function $\nu^*(b,\epsilon,\nu_1)$ such that $F[b,\epsilon,\nu_1,\nu^*(b,\epsilon,\nu_1)]=0$ for sufficiently small ϵ and

$$\nu^*(b,0,\nu_1)=-\bar{J}_1(b,\nu_1)/\bar{J}_0(b,\nu_1)$$

Having shown the existence of a limit cycle by IFT, we proceed to show that the limit cycle is unique for a given value of ν_1 and ν_2 by demonstrating that ν^* is monotonic in b. However, it will be more convenient to employ in place of b another parameter h, which corresponds to the energy level, i.e.,

$$h=H(b,0)=-\nu_1(b^2/2)-(b^3/3)$$

This change of parameter is justified, since $\mathrm{d}h/\mathrm{d}b=-b(\nu_1+b)>0$ for $-\nu_1<b<0$. Thus

$$\nu_2=-J_1(h)/J_0(h)=-P(h) \tag{14}$$

where $J_0(h)=\bar{J}_0[b(h),\nu_1]$, $J_1(h)=\bar{J}_1[b(h),\nu_1]$, and the dependence of ν_1 is suppressed. Since $z_2[b(h)]=z_2[c(h)]=0$, it can be verified that

$$J_0'(h)=\int_{b(h)}^{c(h)}\frac{\mathrm{d}z_1}{z_2}, \qquad J_1'(h)=\int_{b(h)}^{c(h)}\frac{z_1}{z_2}\,\mathrm{d}z_1$$

Furthermore, the limits

$$\lim_{h\to 0}P(h)=-(6/7)\nu_1, \qquad \lim_{h\to-\nu_1^3/6}P(h)=\lim_{h\to-\nu_1^3/6}\frac{J_1'(h)}{J_0'(h)}=-\nu_1$$

agree with the previous calculations of saddle-loop and Hopf bifurcations. The following relationships between $J_0(h)$, $J_1(h)$, and their derivatives can be obtained using the expression for z_2:

$$J_0(h)=\nu_1^2\,J_1'(h)=\int_{b(h)}^{c(h)}\frac{z_1^3}{z_2}\,\mathrm{d}z_1 \tag{15a}$$

$$5J_0(h)-6h\,J_0'(h)+\nu_1^2 J_1'(h)=0 \tag{15b}$$

$$35J_1(h)+6[h\nu_1 J_0'(h)-(\nu_1^3+5h)\,J_1'(h)]=0 \tag{15c}$$

$$(\nu_1^3+6h)\,J_1''(h)=J_0'(h)\nu_1+J_1'(h) \tag{15d}$$

$$6h(\nu_1^3+6h)\,J_0''=\nu_1^2 J_1'(h)-6h\,J_0'(h) \tag{15e}$$

Now, using the preceding relations, one can show that if $P'(h_1)=0$ for some $h_1\ \epsilon(-\nu_1^3/6,0)$ then

$$6h_1(\nu_1^3+6h_1)\frac{P''(h_1)\,J_0(h_1)}{J_0'(h_1)}=-\left\{\nu_1[P(h_1)]-\frac{6h_1}{\nu_1}\right\}^2$$

$$+\frac{6h_1}{\nu_1^2}(\nu_1^3+6h_1)<0 \tag{16a}$$

$$7\nu_1^2P^2(h_1)+6(\nu_1^3-2h_1)\,P(h_1)-6h_1\nu_1=0 \tag{16b}$$

Since $6h_1\,(\nu_1^3+6h_1)<0$ and $J_0(h_1)/J_0'(h_1)<0$, it follows from the inequality (16a) that $P''(h_1)>0$. Furthermore, it follows from Eq. (16b) that $-\nu_1<P(h_1)<0$. In other words, if there is a point h_1 for which $p'(h_1)=0$, then the function P is concave up at this point with the value of the function at this point lying between $-\nu_1$ and 0. Since the end points of $p(h_1)$ are at $-\nu_1$ and $-6/7\ \nu_1$, $p'(h)\neq 0$ for $h_1\epsilon(-\nu_1^3/6,0)$, in fact $p'(h)>0$. Thus, $p(h_1)$ is a monotonically increasing function implying a unique limit cycle.

V. Discussion of Results and Conclusion

The results of this analysis are illustrated in Fig. 2, where the space of unfolding parameters is divided into 10 regions indicating the various bifurcations and phase portraits of Eq. (5). In passing from region 1 to region 2 along OS_1, the nontrivial fixed point changes from an unstable node to an unstable focus while the trivial solution remains a saddle node. Along OH_1, the nontrivial fixed point undergoes a Hopf bifurcation giving birth to an unstable limit cycle. It has been shown that this limit cycle is unique and disappears along OL_1

due to a global bifurcation and a saddle loop that passes through the trivial fixed point is produced. The nontrivial fixed point, in passing from region 4 to region 5 along OS_1', changes from a stable focus to a stable node while the trivial fixed point remains a saddle node. Along OT a transcritical bifurcation takes places where an exchange of stability between the trivial and nontrivial fixed points occurs. Finally, in going from region 6 through to region 10 the nontrivial fixed point remains a saddle node while the scenario of bifurcations for the trivial solution is similar to that of the nontrivial fixed point detailed previously and presented in Fig. 2.

In this paper a complete unfolding of a codimension two bifurcation due to a double-zero eigenvalue of the equations of pitching motion of an aircraft was carried out in the vicinity of zero-stiffness derivatives $S(\sigma_c,h_c)=0$, and zero-damping derivative $D(\sigma_c,h_c)=0$. Unfolding of such a singularity will uncover all possible bifurcations that may be present in the vicinity of the singularity, in addition to the results of Hui and Tobak.[1] Even though the problem considered is not rich enough to fully demonstrate the method of unfolding of a codimension two bifurcation point, as most of the local results could have been obtained using methods adopted in Ref. 1, this method, nevertheless, provides the results pertaining to uniqueness of limit cycles and global bifurcations.

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